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Numerical solution of direct geoelectric problems in 3D conducting media

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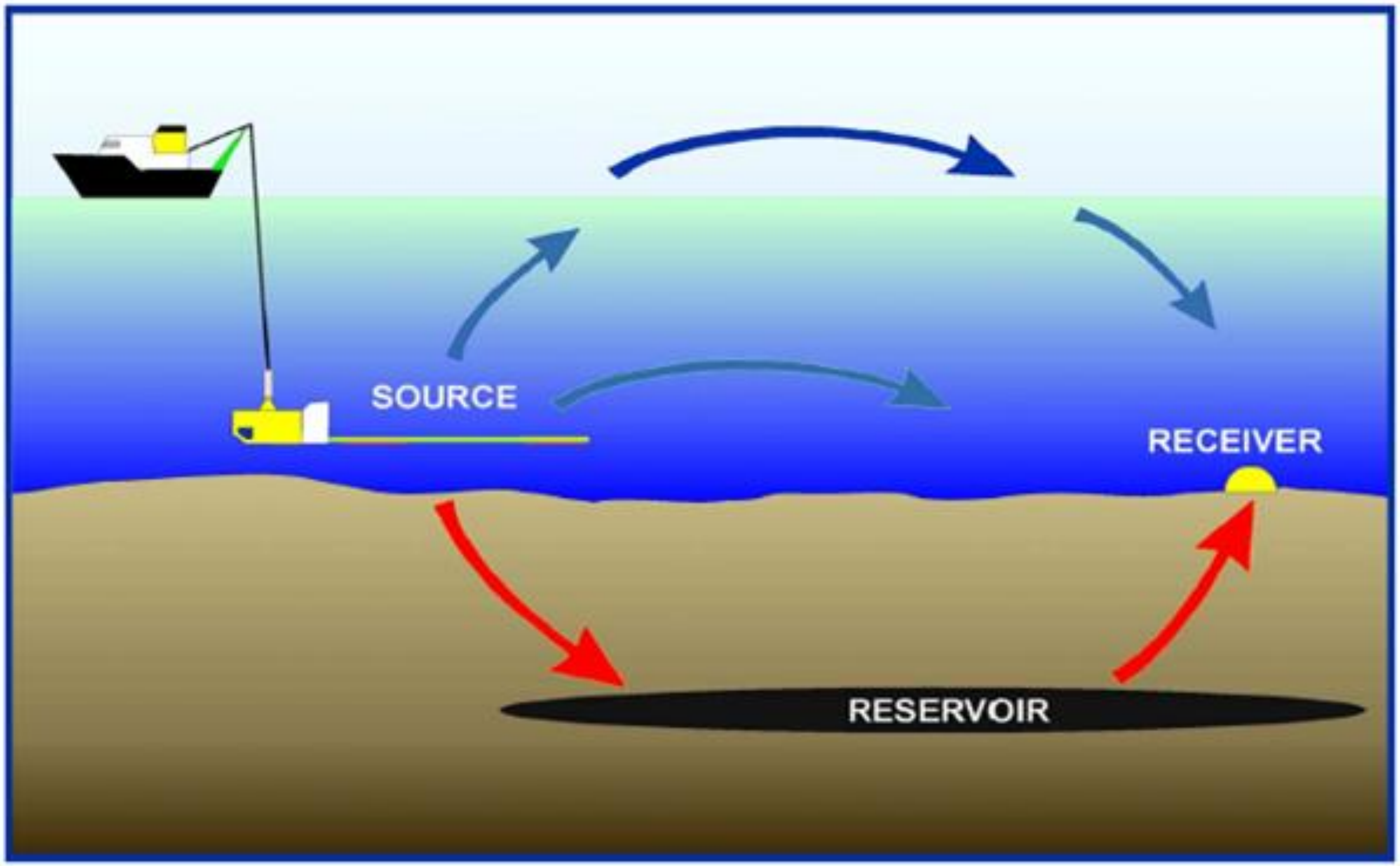
Scheme of electromagnetic survey



Maxwell equations in time domain

$$rot\ \boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{J}^s \qquad div\ (\sigma\boldsymbol{E} + \boldsymbol{J}^s) = 0$$

$$rot\ \boldsymbol{E} + \frac{\partial}{\partial t}(\mu\boldsymbol{H}) = \boldsymbol{0} \qquad div\ \mu\boldsymbol{H} = 0$$

<u>Statement in EM potentials $\boldsymbol{A}$ and U with special gauge condition</u>

$$\mu\boldsymbol{H} = rot\ \boldsymbol{A} \qquad div\ \sigma\boldsymbol{A} = 0$$

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} + \nabla U$$

$$div\ \sigma\nabla U = div\ \boldsymbol{J}^s \qquad \begin{cases} \dfrac{\partial}{\partial t}(\sigma\boldsymbol{A}) + rot\ \dfrac{1}{\mu}\ rot\ \boldsymbol{A} = -\sigma\nabla U + \boldsymbol{J}^s \\ div\ \sigma\boldsymbol{A} = 0 \end{cases}$$

Interface conditions

$$[\boldsymbol{n}\times\boldsymbol{A}]_\Gamma = \boldsymbol{0}, \quad \left[\boldsymbol{n}\times\frac{1}{\mu}rot\ \boldsymbol{A}\right]_\Gamma = \boldsymbol{0}, \quad [\sigma\boldsymbol{A}\cdot\boldsymbol{n}]_\Gamma = 0,$$

$$[U]_\Gamma = 0, \quad [\sigma\nabla U\cdot\boldsymbol{n}]_\Gamma = 0.$$

The main computational difficulties

- Kernel of $\boldsymbol{rot}\,\frac{1}{\mu}\,\boldsymbol{rot}\,(\cdot)$ operator
- Singularity of the solution near the source of the electromagnetic field
- Conditions on the external boundaries

Kernel of $\boldsymbol{rot}\ \frac{\mathbf{1}}{\boldsymbol{\mu}}\ \boldsymbol{rot}\ (\cdot)$ operator

«Redundant» condition

$$\frac{\partial}{\partial t}[div\ \sigma \boldsymbol{A}] = div\left[\frac{\partial}{\partial t}(\sigma \boldsymbol{A}) + rot\ \frac{1}{\mu}\ rot\ \boldsymbol{A}\right] = div[-\sigma \nabla U + \boldsymbol{J}^s] = 0$$

$$div\ \sigma \boldsymbol{A} = div\ \sigma \boldsymbol{A}|_{t=0} = 0$$

<u>Simple statement **(S)**</u>

$$\frac{\partial}{\partial t}(\sigma \boldsymbol{A}) + rot\ \frac{1}{\mu}\ rot\ \boldsymbol{A} = \boldsymbol{F}$$
$$\boldsymbol{A}|_{t=0} = \boldsymbol{A}_0$$

Rough analysis

$$\boldsymbol{A} - \boldsymbol{A}_h = \nabla \varphi \quad and \quad \boldsymbol{F} - \boldsymbol{F}_h = 0$$

$$\frac{\partial}{\partial t}(\sigma \nabla \varphi) = \left[\frac{\partial}{\partial t}(\sigma \boldsymbol{A}) + rot\ \frac{1}{\mu}\ rot\ \boldsymbol{A}\right] - \left[\frac{\partial}{\partial t}(\sigma \boldsymbol{A}_h) + rot\ \frac{1}{\mu}\ rot\ \boldsymbol{A}_h\right] = \boldsymbol{F} - \boldsymbol{F}_h = 0$$

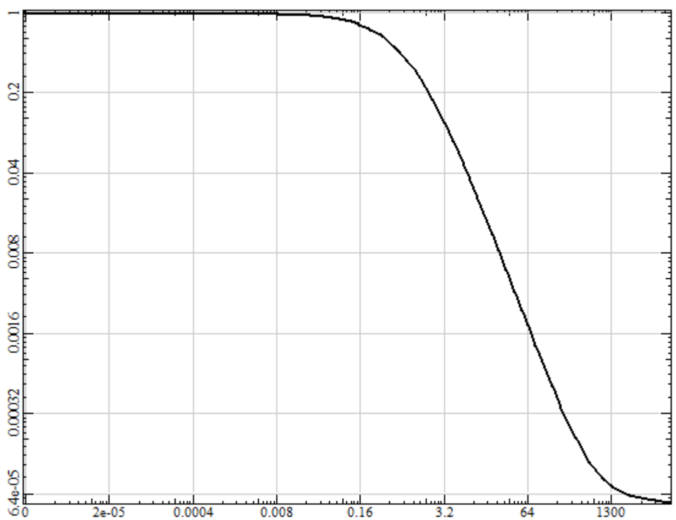
Error is time-independent

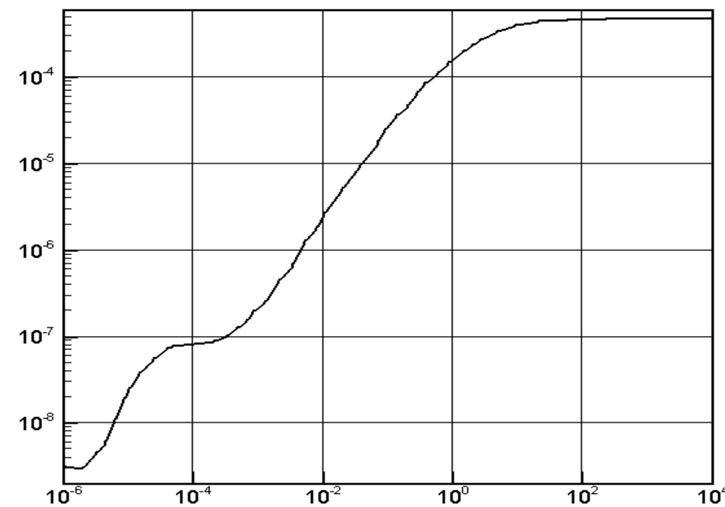
$$\sigma \nabla \varphi = \sigma \nabla \varphi|_{t=0}$$

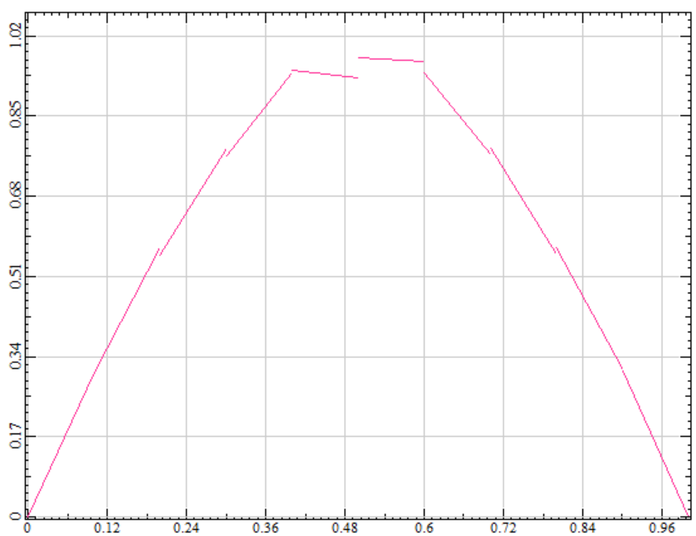
$$\|\boldsymbol{A}(t)\|_{t\to\infty} \to 0 \qquad \frac{\|\boldsymbol{A} - \boldsymbol{A}_h\|}{\|\boldsymbol{A}\|} = \frac{\|\nabla \varphi\|}{\|\boldsymbol{A}\|}_{t\to\infty} \to \infty$$

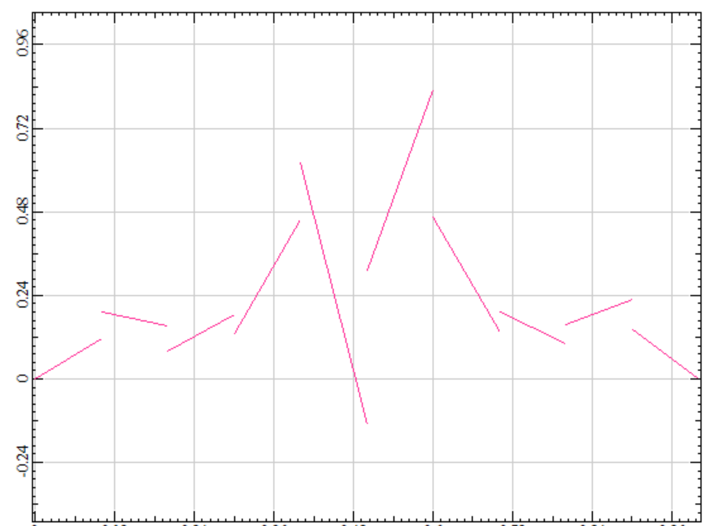
Example in statement **(S)**

$$\Omega = [0,1]^3 \qquad \sigma = 100 \qquad \mu = 4\pi \cdot 10^{-7} \qquad \mathbf{u} = \begin{Bmatrix} sin\pi y \cdot sin\pi z \\ sin\pi x \cdot sin\pi z \\ sin\pi x \cdot sin\pi y \end{Bmatrix} (t+1)^{-1.5}$$









Findings

- Choice of numerical method for solving a simple statement (S) does not solve the “problem of the kernel”
- The simple statement (S) suits only for solutions at early times
- Gauge condition (constraint) is essential and should be included in the statement of the problem

Saddle point problem

$$\begin{cases} \dfrac{\partial}{\partial t}(\sigma \boldsymbol{A}) + rot\ \dfrac{1}{\mu}\ rot\ \boldsymbol{A} - \sigma \nabla p = \boldsymbol{F} \\ div\ \sigma \boldsymbol{A} = 0 \end{cases}$$

Sobolev spaces

$$\boldsymbol{V} \equiv \boldsymbol{H}_0(rot;\Omega) = \{\boldsymbol{v} \in \boldsymbol{L_2}(\Omega):\quad rot\ \boldsymbol{v} \in \boldsymbol{L_2}(\Omega),\ \ \boldsymbol{n} \times \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\ \}$$

$$Q \equiv H_0^1(\Omega) = \{q \in L_2(\Omega):\quad \nabla q \in \boldsymbol{L_2}(\Omega),\ \ q|_{\partial\Omega} = 0\ \}$$

<u>Statement with Lagrange multiplier **(L)**</u>

Find $\quad \boldsymbol{u} \in \boldsymbol{L}_2(0,T;\boldsymbol{V})\ \ p \in \boldsymbol{L}_2(0,T;Q)$

$$\begin{cases} \dfrac{d(M\boldsymbol{u})}{dt} + A\boldsymbol{u} + B^T p = \boldsymbol{F} \\ B\boldsymbol{u} = 0 \end{cases}$$
$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \quad (div\ \sigma \boldsymbol{u}_0 = 0)$$

Where

$$A: \boldsymbol{L}_2(0,T;\boldsymbol{V}) \to \boldsymbol{L}_2(0,T;\boldsymbol{V}') \qquad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \frac{1}{\mu} rot\ \boldsymbol{u} \cdot rot\ \boldsymbol{v}\ d\Omega \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

$$B: \boldsymbol{L}_2(0,T;\boldsymbol{V}) \to \boldsymbol{L}_2(0,T;Q') \qquad \langle B\boldsymbol{u}, q\rangle = -\int_\Omega \sigma \boldsymbol{u} \cdot \nabla q\ d\Omega \quad \forall q \in Q$$

$$M: \boldsymbol{L}_2(0,T;\boldsymbol{V}) \to \boldsymbol{L}_2(0,T;\boldsymbol{V}') \qquad \langle M\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \sigma \boldsymbol{u} \cdot \boldsymbol{v}\ d\Omega \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

Regularized problem

$$(\boldsymbol{u},\boldsymbol{v})_{\sigma,rot} = \int_\Omega (rot\ \boldsymbol{u}\cdot rot\ \boldsymbol{v} + \sigma\boldsymbol{u}\cdot\boldsymbol{v})\ d\Omega \quad \forall \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}$$

Then

$$\boldsymbol{V} = Ker\ A \oplus Ker\ B$$

<u>Regularized problem **(R)**</u>

Find $\boldsymbol{u}\in\boldsymbol{L}_2(0,T;\boldsymbol{V})$

$$\frac{d(M\boldsymbol{u})}{dt} + A_\beta\boldsymbol{u} = \boldsymbol{G}$$
$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$$

Where

$$A_\beta: \boldsymbol{L}_2(0,T;\boldsymbol{V}) \to \boldsymbol{L}_2(0,T;\boldsymbol{V}') \qquad \langle A_\beta\boldsymbol{u},\boldsymbol{v}\rangle = \langle A\boldsymbol{u},\boldsymbol{v}\rangle + \frac{1}{\beta}\langle B\boldsymbol{u}, C^{-1}B\boldsymbol{v}\rangle \quad \forall\boldsymbol{v}\in\boldsymbol{V}$$

$$\boldsymbol{G}: \boldsymbol{L}_2(0,T;\boldsymbol{V}) \to \mathbb{R} \qquad \langle\boldsymbol{G},\boldsymbol{v}\rangle = \langle\boldsymbol{F},\boldsymbol{v}\rangle - \langle B\boldsymbol{v}, C^{-1}div\ \boldsymbol{F}\rangle \quad \forall\boldsymbol{v}\in\boldsymbol{V}$$

And

$$C: \boldsymbol{L}_2(0,T;Q) \to \boldsymbol{L}_2(0,T;Q') \qquad \langle Cp,q\rangle = \int_\Omega \sigma\nabla p\cdot\nabla q\ d\Omega \quad \forall q\in Q$$

Essential property

$$\beta > 0 \qquad div\ \sigma\boldsymbol{u} = 0$$

Finite element problem

Finite element spaces

$$\boldsymbol{V}_h \subset \boldsymbol{V} \quad Q_h \subset Q$$

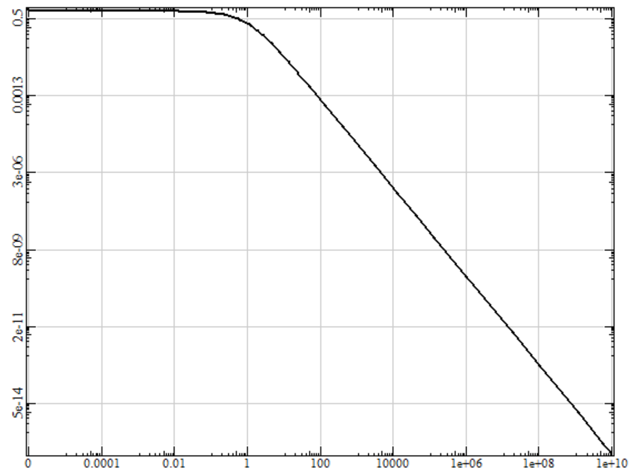
<u>Finite element problem **(RF)**</u>

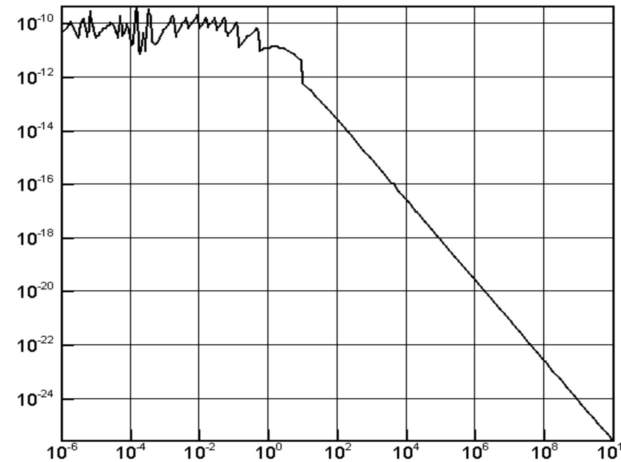
Let $\boldsymbol{u}^0 = \boldsymbol{u}_0$ for $j = 1,2,\dots,N$ find $\boldsymbol{u}^{j+1} \in \boldsymbol{V}_h$

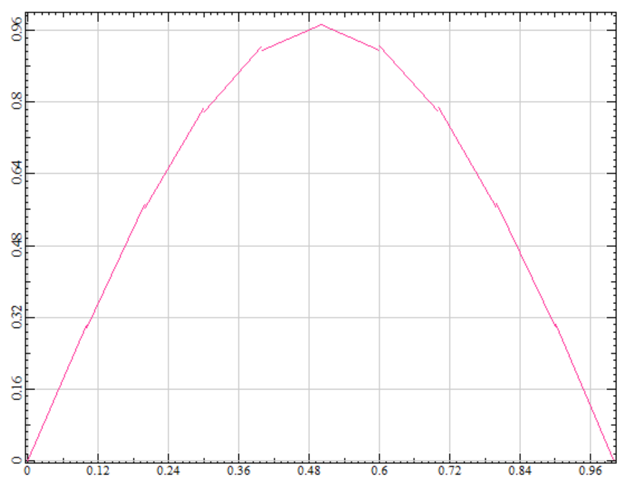
$$\boxed{M\frac{\boldsymbol{u}^{j+1} - \boldsymbol{u}^j}{\tau} + A_\beta \frac{\boldsymbol{u}^{j+1} + \boldsymbol{u}^j}{2} = \boldsymbol{G}^{j+1/2}}$$

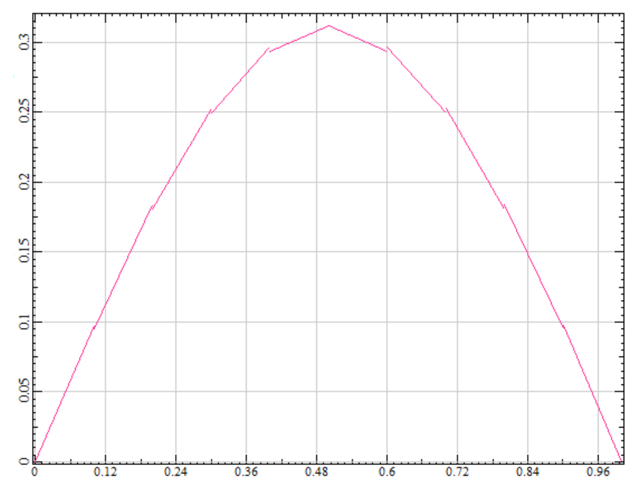
Example in statement **(RF)**

$$\Omega = [0,1]^3 \qquad \sigma = 100 \qquad \mu = 4\pi \cdot 10^{-7} \qquad \mathbf{u} = \begin{Bmatrix} \sin\pi y \cdot \sin\pi z \\ \sin\pi x \cdot \sin\pi z \\ \sin\pi x \cdot \sin\pi y \end{Bmatrix} (t+1)^{-1.5}$$









Singularity of the solution near the source $\boldsymbol{J}^s$

Electric line

$$\boldsymbol{A} = (0,0,z_A), \boldsymbol{B} = (0,0,z_B)$$

$$\boldsymbol{J}^s = (0,0,J_z), \qquad J_z = I\left(\eta(z - z_A) - \eta(z - z_B)\right)\delta(x)\,\delta(y)$$

$$div\,\boldsymbol{J}^s = I\left(\delta(z - z_B) - \delta(z - z_B)\right)\delta(x)\,\delta(y) = I\,\delta(\boldsymbol{r} - \boldsymbol{A}) - I\,\delta(\boldsymbol{r} - \boldsymbol{B})$$

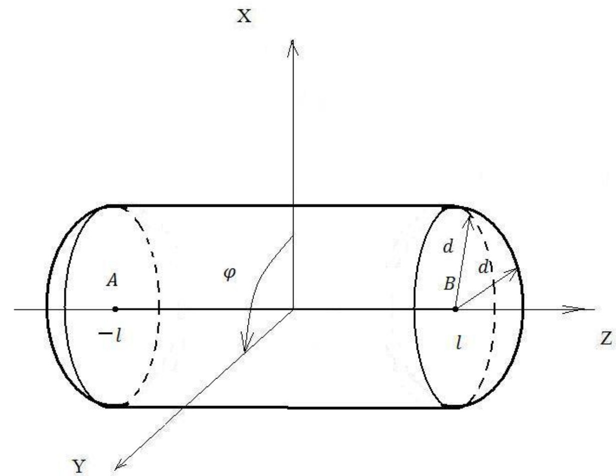
Inhomogeneous constraint

$$div\,\sigma\nabla U = div\,\boldsymbol{J}^s \qquad \begin{cases} \dfrac{\partial}{\partial t}(\sigma \boldsymbol{A}) + rot\,\dfrac{1}{\mu}\,rot\,\boldsymbol{A} = -\sigma\nabla U + \boldsymbol{J}^s \\ div\,\sigma\boldsymbol{A} = g \end{cases}$$

Additive representation

$$\boldsymbol{A} = \boldsymbol{A}^p + \boldsymbol{A}^a \qquad U = U^p + U^a$$

Primary potential U^p in neighborhood $B_{AB}(d) \subset \Omega,\ d \ll 1$



Primary problem

$B_A(d) \subset B_{AB}(d)$	$B_B(d) \subset B_{AB}(d)$
$\begin{cases} div\ \sigma \nabla U^p\vert_{\partial B_A(d)} = I\ \delta(\boldsymbol{r} - \boldsymbol{A}) \\ U^p\vert_{\partial B_A(d)} = 0 \end{cases}$	$\begin{cases} div\ \sigma \nabla U^p\vert_{\partial B_B(d)} = -I\ \delta(\boldsymbol{r} - \boldsymbol{B}) \\ U^p\vert_{\partial B_B(d)} = 0 \end{cases}$

Fundamental solution

$$U^p(r)|_{B_A(d)} = \frac{I}{4\pi\sigma}\left(\frac{1}{r} - \frac{1}{d}\right) \qquad U^p(r)|_{B_B(d)} = -\frac{I}{4\pi\sigma}\left(\frac{1}{r} - \frac{1}{d}\right)$$

Primary potential $\boldsymbol{A}^p$ in neighborhood $B_{AB}(d) \subset \Omega,\ d \ll 1$

Primary problem

$$\begin{cases} rot\,\frac{1}{\mu} rot\,\boldsymbol{A}^p = -\sigma \nabla U^p + \boldsymbol{J}^s \\ div\,\sigma \boldsymbol{A}^p = g \end{cases} \qquad \boldsymbol{n} \times \boldsymbol{A}^p|_{\partial B_{AB}(d)} = \boldsymbol{0}$$

Biot-Savart law

$$\boldsymbol{A}^p(\boldsymbol{r}) = \frac{\mu}{4\pi}\left(\int_{B_A(d)} \frac{-\sigma\nabla U^p(\boldsymbol{\xi})d\boldsymbol{\xi}}{|\boldsymbol{r}-\boldsymbol{\xi}|} + \int_{B_B(d)} \frac{-\sigma\nabla U^p(\boldsymbol{\xi})d\boldsymbol{\xi}}{|\boldsymbol{r}-\boldsymbol{\xi}|} + \int_A^B \frac{\boldsymbol{J}^s(\boldsymbol{\xi})d\boldsymbol{\xi}}{|\boldsymbol{r}-\boldsymbol{\xi}|}\right) - \boldsymbol{f}_{AB}(\boldsymbol{r})$$

$$= \boldsymbol{A}_A^p(\boldsymbol{r}) + \boldsymbol{A}_B^p(\boldsymbol{r}) + \boldsymbol{A}_{AB}^p(\boldsymbol{r}) - \boldsymbol{f}_{AB}(\boldsymbol{r})$$

Definition

$$g = div\,\sigma \boldsymbol{A}^p$$

Primary potential $\boldsymbol{A}^p$ in neighborhood $B_{AB}(d) \subset \Omega,\ d \ll 1$

Let

$$\boldsymbol{A} = (0,0,-l) \qquad \boldsymbol{B} = (0,0,l)$$

Then

$$\boldsymbol{A}_A^p(\boldsymbol{r}) = \left(A_\rho^p(\boldsymbol{r}), 0, 0\right) \qquad A_\rho^p(\boldsymbol{r}) = -\frac{\mu I}{4\pi}\left(1 + \ln\frac{d}{r}\right)$$

$$\boldsymbol{A}_B^p(\boldsymbol{r}) = \left(A_\rho^p(\boldsymbol{r}), 0, 0\right) \qquad A_\rho^p(\boldsymbol{r}) = \frac{\mu I}{4\pi}\left(1 + \ln\frac{d}{r}\right)$$

$$\boldsymbol{A}_{AB}^p(\boldsymbol{r}) = \left(0, 0, A_z^p(\boldsymbol{r})\right) \qquad A_z^p(\boldsymbol{r}) = \frac{\mu I}{4\pi}\ln\left(\frac{z + l + \sqrt{r^2 + (z+l)^2}}{z - l + \sqrt{r^2 + (z-l)^2}}\right)$$

$$\boldsymbol{f}_{AB}(\boldsymbol{r}) = \left(0, 0, f_z^A(\boldsymbol{r})\right), \quad f_z^A = \begin{cases} \dfrac{\mu I}{4\pi}\ln\left(\dfrac{z + l + d}{z - l + \sqrt{d^2 - 4zl}}\right) & z \in [-l-d, -l[, \\ \dfrac{\mu I}{4\pi}\ln\left(\dfrac{z + l + \sqrt{d^2 + (z+l)^2}}{z - l + \sqrt{d^2 + (z-l)^2}}\right) & z \in [-l, l], \\ \dfrac{\mu I}{4\pi}\ln\left(\dfrac{z + l + \sqrt{d^2 + 4zl}}{z - l + d}\right) & z \in]l, l+d]. \end{cases}$$

Anomalous potential $\boldsymbol{A}^a$ in Ω

$$\sigma\nabla U^p \cdot \boldsymbol{n}_d|_{\partial B_A(d)} = -\frac{I}{4\pi d^2} \qquad \sigma\nabla U^p \cdot \boldsymbol{n}_d|_{\partial B_B(d)} = \frac{I}{4\pi d^2}$$

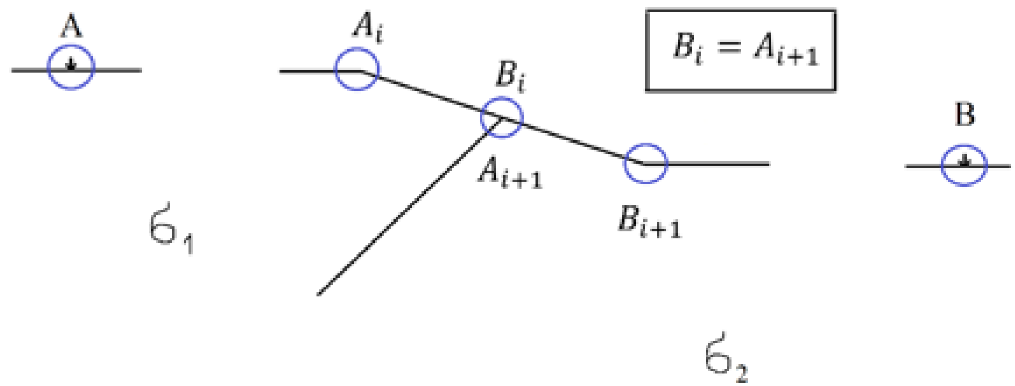
<u>Anomalous scalar problem</u>

$$\begin{gathered} div\ \sigma\nabla U^a = 0 \\ \sigma\nabla U^a \cdot \boldsymbol{n}_d|_{\partial B_{AB}(d)} = \sigma\nabla U^p \cdot \boldsymbol{n}_d|_{\partial B_{AB}(d)} \end{gathered}$$

<u>Anomalous vector problem</u>

$$\begin{cases} \dfrac{\partial}{\partial t}(\sigma \boldsymbol{A}^a) + rot\ \dfrac{1}{\mu}\ rot\ \boldsymbol{A}^a = -\sigma\nabla U^a \\ div\ \sigma\boldsymbol{A}^a = 0 \end{cases}$$

Case with a complex source



$$\boldsymbol{A}^p_{A_{i+1}}(\boldsymbol{r}) = -\boldsymbol{A}^p_{B_i}(\boldsymbol{r})$$

$$U^p = \sum_i U_i^p, \qquad \boldsymbol{A}^p = \sum_i \boldsymbol{A}_i^p.$$

Numerical experiments

$$\Omega = 25\,600\;m \times 25\,600\;m \times 19\,000\;m \qquad \rho = 1\;om \cdot m \qquad |AB| = 500\;m \qquad I_{AB} = 1A$$

Table 1. Meshes

Name	h, m	N_{edges}	N_{nodes}	error, %
Big	125	705 908	116 045	17
Big+	62,5	1 009 552	160 569	3
Big++	31,25	1 221 836	191 301	1

Table 2. Errors ($d = 0.01\;m \div 10.0\;m$)

Time, s	Analytical solution	Big		Big+		Big++	
3,55E-06	1,61E-06	1,33E-06	-17%	1,57E-06	-3%	1,60E-06	-1%
2,82E-05	1,61E-06	1,34E-06	-17%	1,57E-06	-3%	1,60E-06	-1%
2,82E-04	1,61E-06	1,39E-06	-14%	1,57E-06	-2%	1,60E-06	-1%
2,24E-03	1,61E-06	1,48E-06	-8%	1,57E-06	-3%	1,60E-06	-1%
1,78E-02	1,09E-06	1,10E-06	1%	1,10E-06	1%	1,09E-06	0%
1,41E-01	1,64E-07	1,63E-07	-1%	1,63E-07	-1%	1,63E-07	-1%

Conclusion

- Conditions on the external boundaries

Current solution

$$diam\, \Omega > (50 \div 100) \times diam\, (supp\, \boldsymbol{J}^s)$$

Promising solution

Use of elements with suitable conditions at infinity (Infinite elements)

Papers available online

1. I. A. Kremer and M. V. Urev A regularization method for stationary Maxwell equations in an inhomogeneous conducting medium //Numerical Analysis and Applications, 2009, Volume 2, Number 2, P.131-139
2. I. A. Kremer and M. V. Urev Solution of a regularized problem for a stationary magnetic field in a nonhomogeneous conducting medium by a finite element method //Numerical Analysis and Applications, 2010, Volume 3, Number 1, P.25-38
3. M. I. Ivanov, I. A. Kremer, and M. V. Urev Regularization Method for Solving the Quasi-Stationary Maxwell Equations in an Inhomogeneous Conducting Medium //Computational Mathematics and Mathematical Physics, 2012, Vol. 52, No. 3, pp. 476–488.