Helmholtz Equation in Highly Heterogeneous Media

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Heterogeneous Helmholtz equation

- the pressure u satisfies the equation

$$\begin{cases} -\dfrac{\omega^2}{c^2}u - \Delta u = f & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - \dfrac{\mathbf{i}\omega}{c}u = 0 & \text{on } \partial\Omega \end{cases}$$

- where f is a source of excitation
- where ω is the angular frequency
- where c is the wave velocity

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Frequency

- the frequency is represented by the parameter $\omega \in \mathbb{R}_+^*$
- solving for high frequencies requires heavy computations
- high order methods may reduce the computational cost
- coarse mesh with high order elements

Heterogeneous Helmholtz equation

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Wave velocity

- the wave velocity is represented as a function $c : \Omega \to \mathbb{R}_+^*$
- in geophysical applications, c is piecewise constant
- in classical FEM, the mesh has to fit c
- c must be constant in each cell of the mesh
- for highly heterogenous media, we need a fine mesh

Stability Analysis

- Bound the norm of u
- Explicitly in terms of f, ω and c

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Homogeneous case

- In star-shaped domain

$$||u|| \leq C\left(\frac{\omega}{c}\right)^{-1}||f||, \quad ||\nabla u|| \leq C||f||, \quad ||\nabla^2 u|| \leq C\frac{\omega}{c}||f||,$$

  with $C := C(\Omega)$
- Proof with $v = \mathbf{x} \cdot \nabla u$
- See Melenk [1995], Hetmaniuk[2007]

One dimentional monotonous case

- If c is non-increasing (and piecewise e.g. C^1)

$$||u|| \le C_s \omega^{-1} ||f||, \quad ||\nabla u|| \le C_s ||f||, \quad ||\nabla^2 u|| \le C_{s,2} \omega ||f||,$$

  with $C_s = 2Lc_{max}/c_{min}$ and $C_{s,2} = 1 + c_{max} C_s$.
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One dimentional C^1 case

- If $c \in C^1$

$$||u|| \le C_s \omega^{-1} ||f||, \quad ||\nabla u|| \le C_s ||f||, \quad ||\nabla^2 u|| \le C_{s,2} \omega ||f||,$$

  with $C_s = C \exp((2|c'_{min}|/c_{min} + 1)L)$ and $C_{s,2} = 1 + c_{max} C_s$.
- Proof with

$$v(x) = \exp\left\{ \left(2\frac{|c'_{min}|}{c_{min}} + 1)z \right) \right\} u'(x)$$

Two dimentional piecewise constant case

- If Ω star-shaped w.r.t. $\mathbf{x}$, and

$$\frac{1}{c_r^2}\mathbf{x}\cdot\mathbf{n}_r + \frac{1}{c_l^2}\mathbf{x}\cdot\mathbf{n}_l \leq 0, \quad \forall x \in \Omega_r \cap \Omega_l, \quad \forall r, l$$

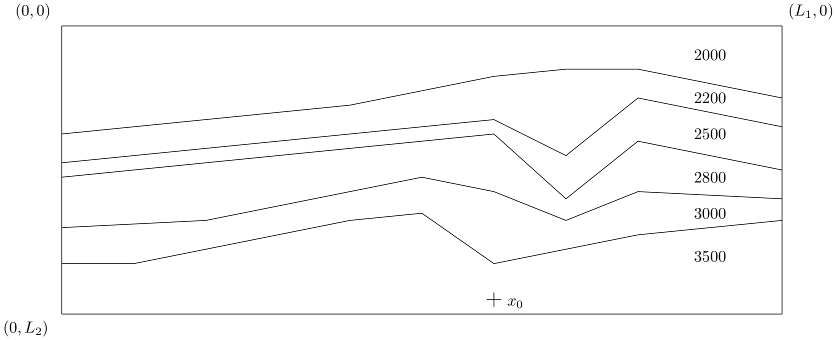
  then

$$||u|| \leq C_s\omega^{-1}||f||, \quad ||\nabla u|| \leq C_s||f||, \quad ||\nabla^2 u|| \leq C_{s,2}\omega||f||,$$

  with $C_s := C(\Omega, c_{max}/c_{min})$, and $C_{s,2} = 1 + c_{max}C_s$.

- Proof with $v = \mathbf{x}\cdot\nabla u$.

A Stratified medium



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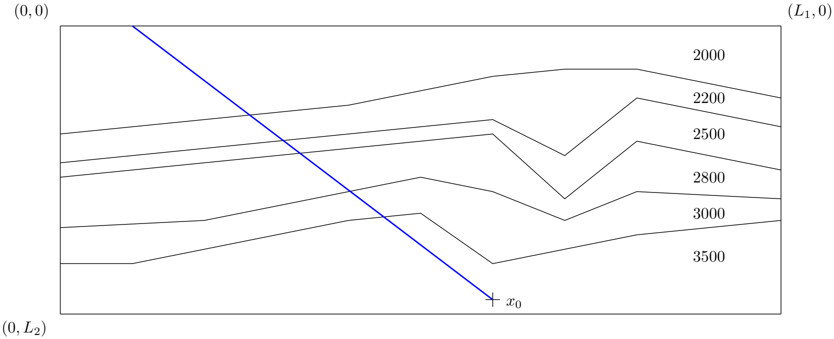

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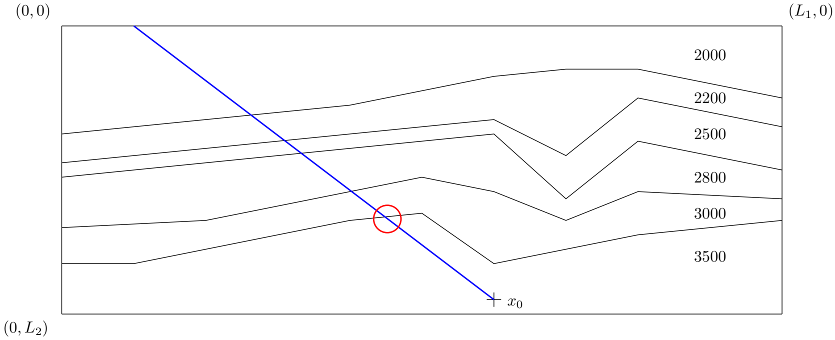

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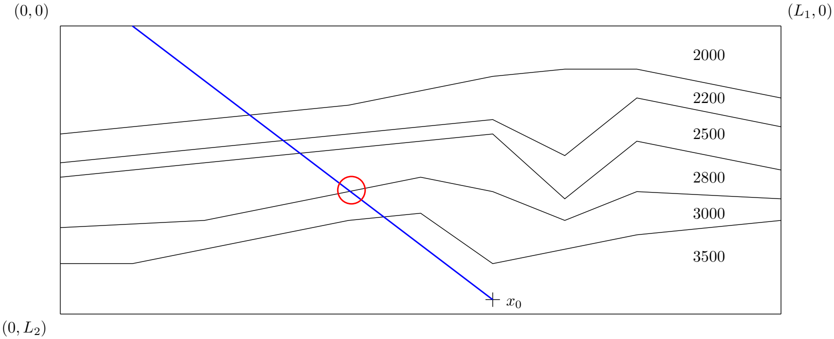

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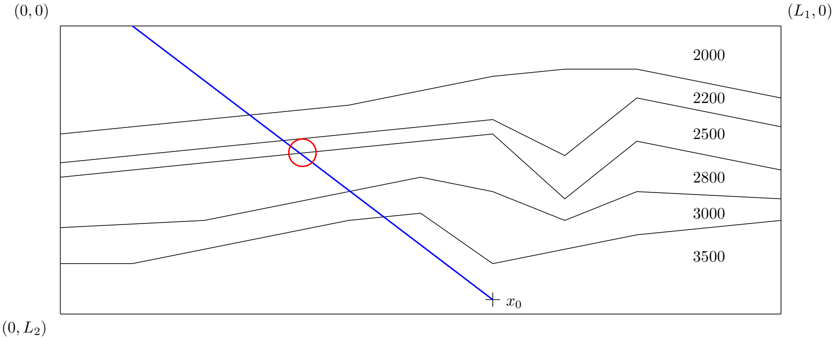

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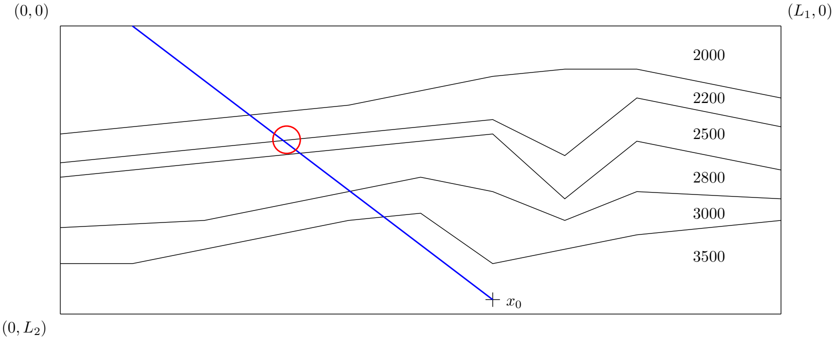

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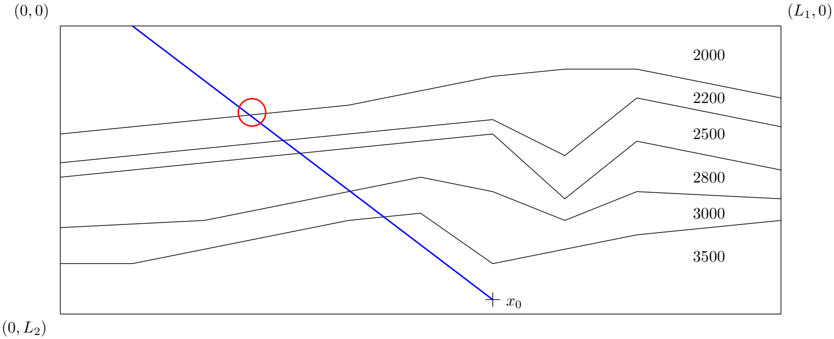
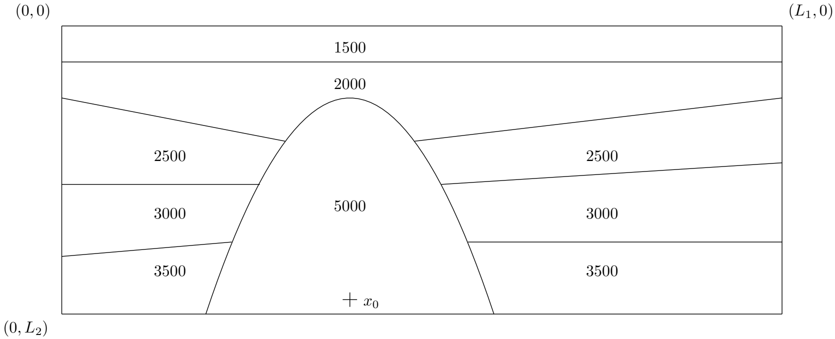


A Salt body



Finite element space

- $\mathcal{T}_H$ mesh of Ω
- H mesh step
- V_H^p discretisation space of order p

$$V_H^p = \left\{ v \in C^0(\bar{\Omega}) \quad | \quad v|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_H \right\}$$

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Linear mapping

- reference cell $\hat{K}$
- Linear map $\mathcal{F}_K : \hat{K} \to K$
- Constant jacobian $\text{jac}(\mathcal{F}_K)$

Basic principle of the FEM

- Transform the PDE in a linear system $MU = F$
- $\dim M$ related to H and p
- The entries involve quantites of the form

$$\int_K \frac{1}{c^2} \varphi^i \varphi^j,$$

- where $K \in \mathcal{T}_H$ is a cell,
- where $\varphi^i, \varphi^j \in \mathcal{P}_p(K)$ are polynomials

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Goals

- Cheap formula for the entries
- Keep $\dim M$ as small as possible
- Stability and convergence for small H

Computations of the entries

- c is constant

$$\int_K \frac{1}{c^2}\varphi^i\varphi^j = \frac{1}{c^2}\int_K \varphi^i\varphi^j$$

- Linear mapping

$$\int_K \varphi^i\varphi^j = \text{jac}(\mathcal{F}_K)\int_{\hat{K}} \hat{\varphi}^i\hat{\varphi}^j = \text{jac}(\mathcal{F}_K)\hat{M}_{ij}$$

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Numerical Implementation

- Store once the references values

$$\hat{M}_{ij} = \int_{\hat{K}} \hat{\varphi}^i\hat{\varphi}^j$$

- Weight the reference value on each cell by the jacobian and c

$$\frac{1}{c^2}\mathrm{jac}(\mathcal{F}_K)\hat{M}_{ij}$$

Stability and convergence (Melenk Sauter [2010])

- A (non-optimal) condition for stability $\omega^{p+1} H^p \leq C$
- Convergence $|u - u_H|_{1,\Omega} \leq C\omega H$
- Proof based on best-approximation properties

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One dimentional case (Babuska Ihlenburg [1997])

- A (optimal) condition for stability $\omega^{2p+1}H^{2p} \leq C$
- Convergence $|u - u_H|_{1,\Omega} \leq C\omega H$
- Proof using the numerical Green function for uniform mesh

The pollution effect

- Consider $p = 1$, then $\omega^2 H \leq C$ and $H \leq C\omega^{-2}$
- $2 \times \omega$ implies $H/4$

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High order reduce pollution

- Consider $p = 5$, then $\omega^6 H^5 \leq C$ and $H \leq C\omega^{-5/4}$
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Conclusion concerning pollution

- Coarse cells
- High order elements
- Smaller dim M for a given precision

Computations of the entries on fine meshes

- c is not constant on Ω
- c is constant on each cell K and

$$\int_K \frac{1}{c^2}\varphi^i\varphi^j = \frac{1}{c^2}\int_K \varphi^i\varphi^j$$

- Requires fine cells
- Not adapted to handle pollution

Coarse meshes

- No analytical expression of

$$\int_K \frac{1}{c^2} \varphi^i \varphi^j$$

- Idea: replace c by a simpler parameter c_h

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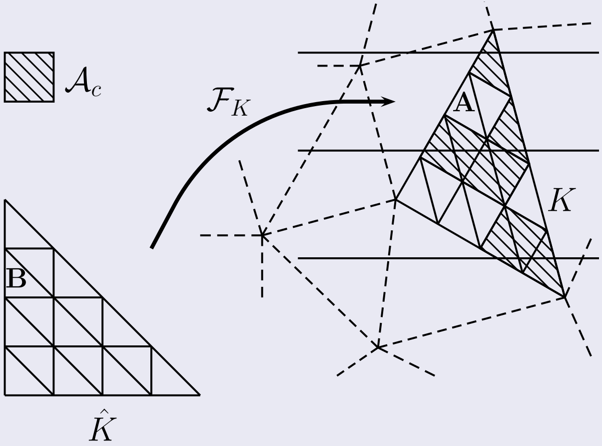
Construction of c_h

- Consider a submesh $\hat{\mathcal{T}}_h$ of $\hat{K}$
- On K, map $\hat{\mathcal{T}}_h$ to $\mathcal{T}_h^K$
- Take c_h to be constant on each subcell of $\mathcal{T}_h^K$

Mapping



Computations of the entries

- c_h is constant on the subcells

$$\int_K \frac{1}{c_h^2}\varphi^i\varphi^j = \sum_{A\in\mathcal{T}_h^K} \frac{1}{c_h^2}\int_A \varphi^i\varphi^j$$

- Linear mapping

$$\begin{aligned}\sum_{A\in\mathcal{T}_h^K} \frac{1}{c_h^2}\int_A \varphi^i\varphi^j &= \mathrm{jac}(J_K)\sum_{B\in\hat{\mathcal{T}}_h} \frac{1}{c_h^2}\int_B \hat{\varphi}^i\hat{\varphi}^j \\ &= \mathrm{jac}(J_K)\sum_{B\in\hat{\mathcal{T}}_h} \frac{1}{c_h^2}\hat{M}_{ij}^B\end{aligned}$$

Numerical Implementation

- Store once the references values

$$\hat{M}_{ij}^{B} = \int_{B} \hat{\varphi}^{i} \hat{\varphi}^{j} \quad \forall B \in \hat{\mathcal{T}}_h$$

- Weight the references values with c_h and the jacobian

$$\text{jac}(\mathcal{F}_K) \sum_{B \in \hat{\mathcal{T}}_h} \frac{1}{c_h^2} \hat{M}_{ij}^{B}$$

Stability and convergence

- Extension of the homogeneous result for $p = 1$
- A (non-optimal) stability condition: $\omega^2 H + \omega h \leq C$
- Convergence: $|u - u_{H,h}|_{1,\Omega} \leq C\,(\omega H + \omega h)$

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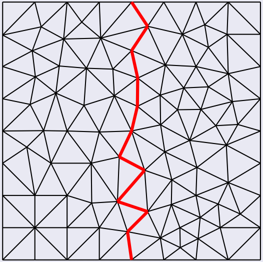
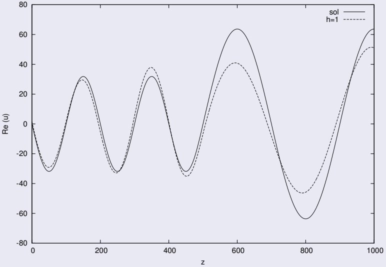
One dimentional case

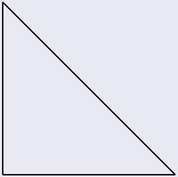
- Extension of the homogeneous result for $p = 1, 2, 3$.
- A (non-optimal) stability condition: $\omega^{p+1} H^p < C$
- Convergence: $|u - u_{H,h}|_{1,\Omega} \leq C(\omega H + \omega h)$

Two-layer medium ($\omega = 10\pi$, $c_1 = 1000$, $c_2 = 2000$, $\mathcal{P}_4$ elements)





Reference cell



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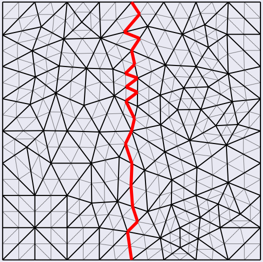

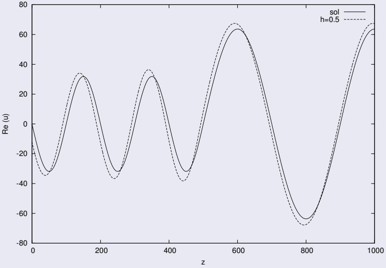

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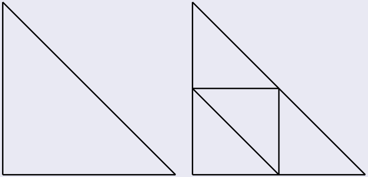

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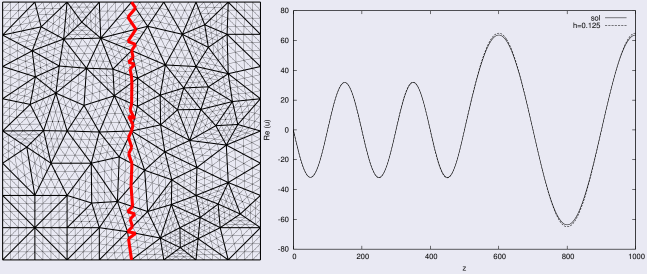

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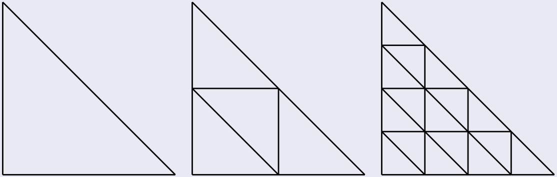

Multi-layer test case

- 1000 layers of 3 meters each
- $c_{min} = 500 m.s^{-1}$, $c_{max} = 5500 m.s^{-1}$
- $|c_j - c_{j+1}| \geq 1000$
- $\omega = 40\pi$ ($f = 20$ Hz)
- $\mathcal{P}_6$ elements

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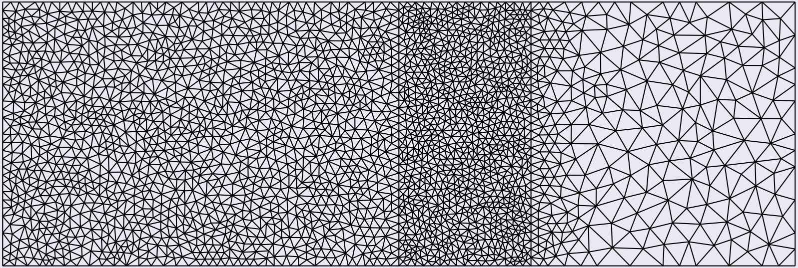
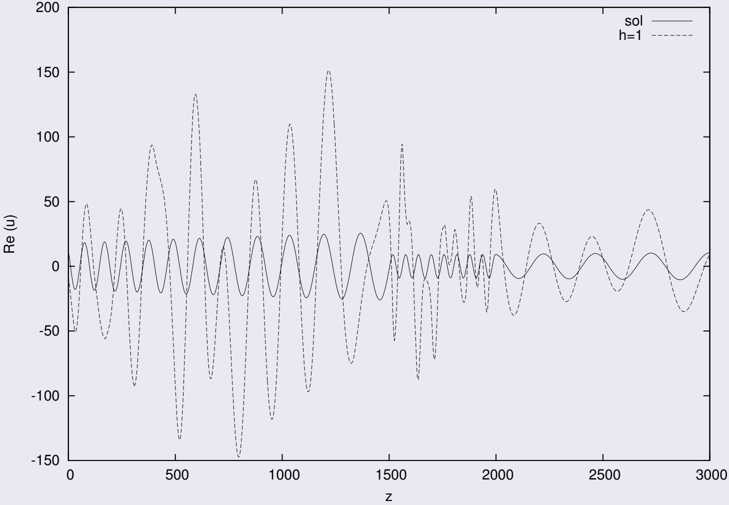
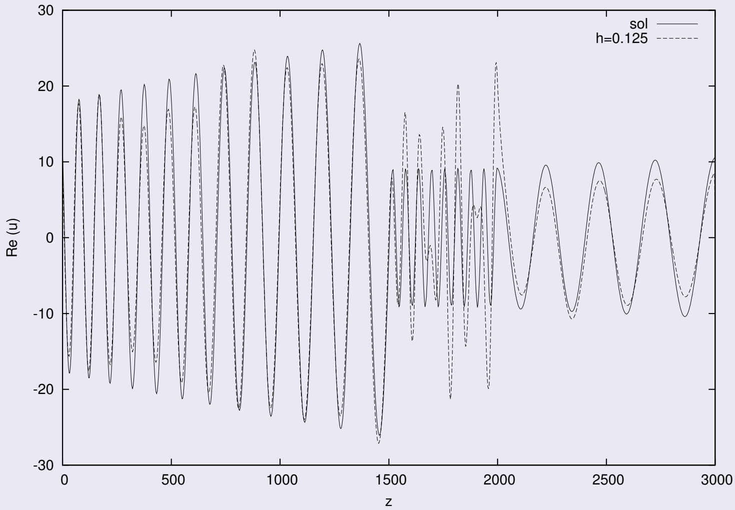
Mesh



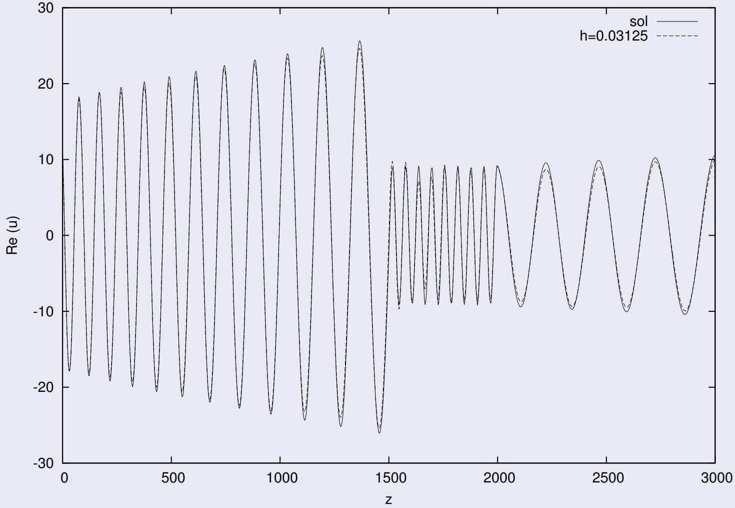
Without subquadrature (744% relative L^2 error)



64 subcells (44.5% relative L^2 error)



1024 subcells (7.00% relative L^2 error)



Test case

- Marmousi velocity model
- 5 Hz
- 10 shots at 100 m detph

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Coarse mesh

- 4743 cells
- $\mathcal{P}_6$ elements
- without subcells
- 16 subcells

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- 5 Hz
- 10 shots at 100 m detph

Coarse mesh

- 4743 cells
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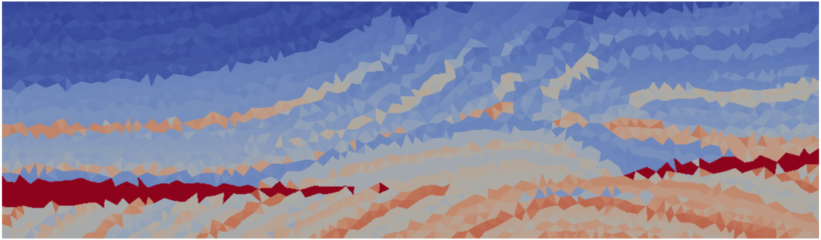
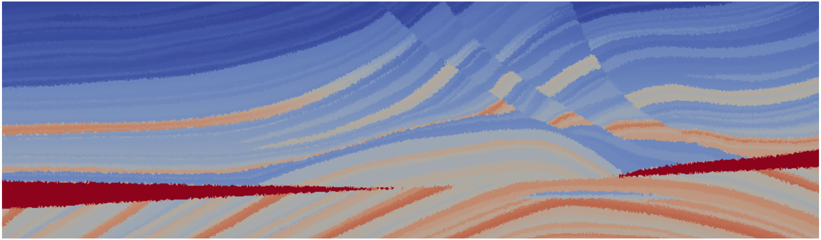
Fine mesh

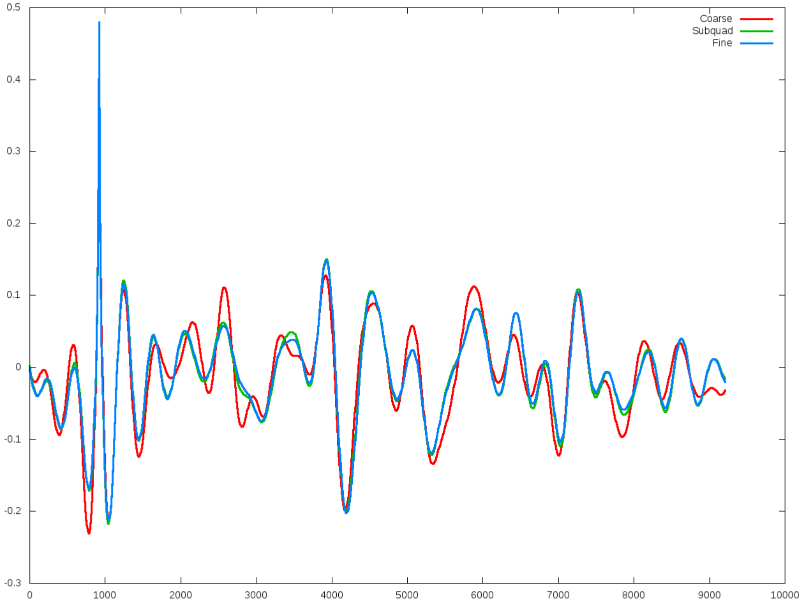
- 47531 cells
- $\mathcal{P}_6$ elements
- without subcells

Approximated velocity model c_h



Approximated velocity model c_h (16 subcells)




Coarse
Subquad
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0.5
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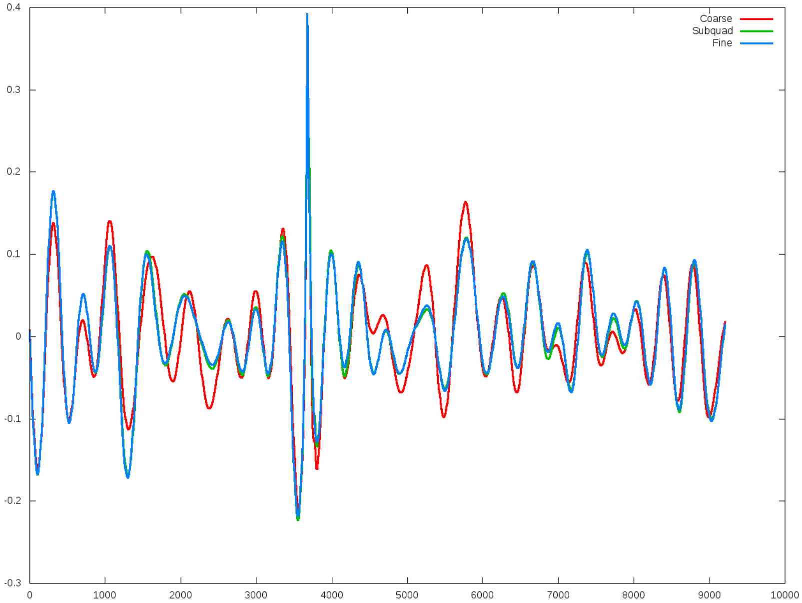
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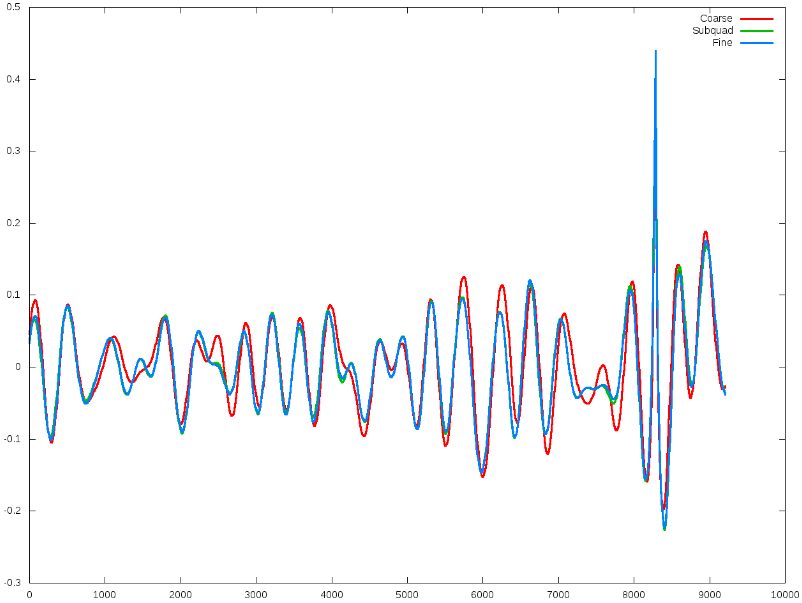
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Conclusion

- subquadrature schemes capture fine scale heterogenities
- an arbitrary high order 2D solver has been implemented
- analytical study of the PDE
- theoretical convergence issues have been adressed
- but the estimates are not sharp

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Ongoing work

- 3D solver
- sharper estimates
- non-constant density

$$-\frac{\omega^2}{\kappa}u - \mathrm{div}\left(\frac{1}{\rho}\nabla u\right) = f$$

Thank you for your attention!