On the variational ADI methods

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$$Au = f, \ \ A \in \mathcal{R}^{N,N}, \ \ u, f \in \mathcal{R}^{N},$$
$$A = A_1 + A_2, \ \ A_1 A_2 = A_2 A_1,$$
$$(A_l v, v) \geq \gamma_l (v, v), \ \ \gamma_l > 0, \ \ l = 1, 2,$$
$$\Omega^h = \{x = x_i = ih_x, \ y = y_j = jh_y, \ i = 1, ..., N_1; \ j = 1, ..., N_2\},$$
$$u_k = u_{i,j}, \ \ f_k = f_{i,j}, \ \ k = (i-1)N_2 + j, \ \ k = 1, ..., N_1 N_2,$$
$$A_1 = \textbf{block-diag}\{\bar{A}_1\}, \ \bar{A}_1 \in \mathcal{R}^{N_1,N_1},$$
$$(A_1 u)_{i,j} = h_x^{-2}(-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}),$$
$$\{u_{i,j}\} = U \in \mathcal{R}^{N_1,N_2}, \ \ \{f_{i,j}\} = F \in \mathcal{R}^{N_1,N_2}$$

$$U\bar{A}_1 + \bar{A}_2 U = F, \quad \bar{A}_2 \in \mathcal{R}^{N_2,N_2},$$
$$(\bar{A}_2 u)_{i,j} = h_y^{-2}(-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}),$$
$$I_{N_1} \otimes \bar{A}_1 + (\bar{A}_2)^T \otimes I_{N_2} = f,$$
$$u^{n+1/2} = u^n + \alpha_n(f - A_1 u^{n+1/2} - A_2 u^n),$$
$$u^{n+1} = u^{n+1/2} + \beta_n(f - A_1 u^{n+1/2} - A_2 u^{n+1}),$$
$$u^{n+1} = u^n + (\alpha_n + \beta_n)(I + \beta_n A_2)^{-1}(I + \alpha_n A_1^{-1})(f - Au^n),$$
$$B_n(u^{n+1} - u^n) = r^n \equiv f - Au^n,$$
$$B_n = (\alpha_n + \beta_n)^{-1}(I + \alpha_n A_1)(I + \beta_n A_2)$$

$$r^{n+1} = r^n - AB_n^{-1} r^n = T_n r^n,$$

$$T_n = I - AB_n^{-1} = (B_n - A)B_n^{-1} =$$

$$= (I - \beta_n A_1)(I - \alpha_n A_2)(I + \alpha_n A_1)^{-1}(I + \beta_n A_2)^{-1},$$

$$T_n = T_n^{(1)} T_n^{(2)},$$

$$T_n^{(1)} = (I - \beta_n A_1)(I + \alpha_n A_1)^{-1},$$

$$T_n^{(2)} = (I - \alpha_n A_2)(I + \beta_n A_2)^{-1},$$

$$r^{n+1} = T_n T_{n-1} \cdots T_0 \; r^0 = \bar{T}_n r^0,$$

$$\bar{T}_n = \bar{T}_n^{(1)} \bar{T}_n^{(2)}, \quad \bar{T}_n^{(l)} = \prod_{k=0}^{n} T_k^{(l)}, \quad l = 1, 2$$

$$Au(t) + \frac{d\,u}{d\,t} = 0, \qquad u(0) = f,$$

$$u(t) = e^{-tA} f,$$

$$\int\limits_0^\infty u(t) d\,t = -A^{-1} e^{-tA}\Big|_0^\infty \cdot f = A^{-1} f,$$

$$F = \hat{F}\check{F}, \quad \hat{F} \in \mathcal{R}^{N_1,m}, \quad \check{F} \in \mathcal{R}^{m,N_2}, \quad m < \min(N_1, N_2),$$

$$\hat{u} = e^{-tA_1} f \approx \sum_{i=1}^{m_0} y_i e^{-t\theta_i},$$

$$u \approx \check{u} = e^{-tA_2}\hat{u} \approx \sum_{j=1}^{m_i} y_{i,j} \quad \mathbf{exp} \quad (-t\eta_{i,j})$$

$$\alpha_n = \beta_n :$$

$$T_n^{(l)} = (I - \alpha_n A_l)(I + \alpha_n A_l)^{-1}, \ l = 1, 2,$$

$$r^{n+1} = \bar{T}_n^{(2)} \bar{T}_n^{(1)} r^0 \equiv \bar{T}_n^{(2)} \bar{r}_1^n,$$

$$|| T_n^{(l)} ||_2 < 1, \ l = 1, 2, \ |r^{n+1}||_2 < ||\bar{r}_1^n||,$$

$$\mathcal{K}_{n+1}^{(rat)}(r^0, A_1) = \mathrm{span}(r^0, \bar{T}_1^{(1)} r^0, ..., \bar{T}_n^{(1)} r^0),$$

$$T_k^{(1)} = P_k^{(1)} Q_k^{(1)}, \ P_k^{(1)} = I - \alpha_k A_1,$$

$$Q_k^{(1)} = (I + \alpha_k A_1)^{-1}, \ \bar{P}_n^{(1)} = \prod_{k=0}^{n} P_k^{(1)},$$

$$\bar{Q}_n^{(1)} = \prod_{k=0}^{n} Q_k^{(1)},$$

$$r^{n+1} = \bar{T}_n^{(2)} \bar{Q}_n^{(1)} \bar{P}_n^{(1)} r^0 \equiv \bar{T}_n^{(2)} \bar{Q}_n^{(1)} \hat{r}_n^{(1)}, \quad \hat{r}_n^{(1)} = \bar{P}_n^{(1)} r^0,$$

$$||\bar{Q}_n^{(1)}||_2 < 1, \qquad ||r^{n+1}||_2 < ||\hat{r}_{n+1}^{(1)}||_2,$$

$$\hat{r}_{n+1}^{(1)} = \bar{P}_{n+1}^{(1)} r^0 = (I - \alpha_n A_1) \cdots (I - \alpha_0 A_1) r^0,$$

$$\mathcal{K}_{n+1}^{(pol)}(r^0, A_1) = \mathrm{span}(r^0, A_1 r^0, ..., A_1^n r^0),$$

$$||\hat{r}_n^{(1)}||_2 = (\hat{r}_{n+1}^{(1)}, \hat{r}_{n+1}^{(1)})^{1/2} \le \varepsilon_1 ||r^0||_2, \ \varepsilon_1 \ll 1,$$

$$r^0 = f - Au^0,\ u^{n+1} = u^n + \hat{\alpha}_n p^n,\ n = 0, 1, ...,$$

$$\hat{r}^{(1)}_{n+1} = \hat{r}^{(1)}_n - \hat{\alpha}_n A_1 p^n = r^0 - \hat{\alpha}_0 A_1 p^0 - ... - \hat{\alpha}_n A_1 p^n,$$

$$(A_1 p^k, A_1 p^n) = \rho_n \delta_{k,n},\ \rho_n = (A_1 p^n, A_1 p^n),$$

$$||\hat{r}^{(1)}_{n+1}||_2^2 = (r^0, r^0) - \sum_{k=0}^{n} \hat{\alpha}_k [2(r^0, A_1 p^k) - \hat{\alpha}_k (A_1 p^k, A_1 p^k)],$$

$$\frac{\partial ||\hat{r}^{(1)}_{n+1}||_2^2}{\partial \hat{\alpha}_k} = 0, \;\; \hat{\alpha}_k = (r^0, A_1 p^k)/\rho_k, \qquad \rho_k = (A_1 p^k, A_1 p^k),$$

$$p^0 = r^0, \;\; p^{n+1} = \hat{r}^{(1)}_{n+1} + \sum_{k=0}^{n} \hat{\beta}_{n,k} p^k, \quad n = 0, 1, ...$$

$$\hat{\beta}_{n,k} = -(A_1 p^k, A_1 \hat{r}^{(1)}_{n+1})/\rho_k,$$

$$\hat{\alpha}_n = (A_1 \hat{r}^{(1)}_n, \hat{r}^{(1)}_n)/\rho_k,$$

Modified Gram–Schmidt:

$$\check{\beta}_{n,k} = -(A_1 p^k, A_1 r^{n,k})/\rho_k, \quad k = 0, 1, \ldots, n,$$

$$r^{n,k} = r^{n,k-1} + \check{\beta}_{n,k-1} p^{k-1},$$

$$r^{n,n} = p^{n+1}, \quad r^{n,0} = \hat{r}^{(1)}_{n+1},$$

$$\frac{\partial ||r^{n,k}||_2}{\partial \check{\beta}_{n,i}} = 0, \quad i = 0, \ldots, k-1,$$

in exact arithmetics:

$$\check{\beta}_{n,k} = \hat{\beta}_{n,k}, \; n = N_1, \; \hat{r}^{(1)}_n = 0,$$

A_1 **has** $m < N_1$ **different** α_k :

$$\bar{P}^{(1)}_{N_1}(\lambda) = \prod_{k=0}^{m} (\lambda_k - \lambda)^{m_k} / \prod_{k=0}^{m} \lambda_k^{m_k}, \quad \sum_{k=0}^{m} m_k = N_1,$$

$$P^{(1)}_{N_1}(\lambda) = \prod_{k=1}^{N_1/2} [(m_c - \lambda)^2 - \delta_k^2] / \prod_{k=1}^{N_1/2} (m_c^2 - \delta_k^2), \quad \delta_k = m_c - \lambda_k,$$

$m_c = (\lambda_k + \lambda_{N_1+1-k})/2$ **if** N_1 **is even**;

$$P^{(1)}_{N_1(\lambda)} = (\lambda_{\frac{\bar{N}_1}{2}} - \lambda) \prod_{k=0}^{\bar{N}_1/2-1} [(m_c - \lambda)^2 - \delta_k^2] /$$

$$/[\lambda_{\frac{\bar{N}_1}{2}} \prod_{k=0}^{\bar{N}_1/2-1} (m_c^2 - \delta_k^2)], \qquad \textbf{if} \ N_1 \ \textbf{is odd},$$

$$\hat{r}^{(1)}_{\bar{N}_1/2} = 0, \ ||r^{n+1}||_2 \le \varepsilon ||r^0||_2, \ \ \varepsilon \ll 1,$$

$$\bar{P}_0^{(1)}(t) = 1, \;\; \bar{P}_{n+1}^{(1)}(t) = \bar{P}_n^{(1)}(t) - \hat{\alpha}_n R_n^{(1)} t,$$

$$R_0^{(1)}(t) = 1, \;\; R_{n+1}^{(1)}(t) = \bar{P}_{n+1}^{(1)}(t) + \sum_{k=0}^{n} \hat{\beta}_{n,k} R_k^{(1)},$$

$$\hat{\beta}_{n,0} = \hat{\beta}_n = \sigma_n^{(1)}/\sigma_{n-1}^{(1)}, \;\; \sigma_n^{(1)} = (A_1 \hat{r}_n^{(1)}, \hat{r}_n^{(1)}),$$

$$p^{n+1} = r^{n+1} + \hat{\beta}_n p^n, \;\; A_1 p^{n+1} = A_1 r^{n+1} + \hat{\beta}_n A_1 p^n,$$

$$\bar{P}_0^{(1)} = R_0^{(1)} = 1, \;\; \bar{P}_{n+1}^{(1)}(t) = \bar{P}_n^{(1)}(t) - \hat{\alpha}_n t R_n^{(1)},$$

$$R_{n+1}^{(1)}(t) = \bar{P}_{n+1}^{(1)}(t) + \hat{\beta}_n R_n^{(1)}(t), \;\; n = 0, 1, ...,$$

symmetrization:

$$\tilde{A}u \equiv DAu = \tilde{f} = Df, \;\; \tilde{A}_1 = DA = \tilde{A}_1^T,$$

scaling: $\tilde{D}\tilde{A}\tilde{D}\tilde{D}^{-1}u = \tilde{D}\hat{f} \equiv \hat{f}$,

$$\hat{A}\hat{u} = \hat{f},\ \hat{A} = \tilde{D}\tilde{A}\tilde{D} = \{\hat{a}_{i,j}; \hat{a}_{i,i} = 1\},\ \hat{u} = \tilde{D}^{-1}u,$$

$$p^{n+1} = Q_n^{(1)} r^{n+1} + \beta_n p^n,\ A_1 p^{n+1} = A_1 Q_n^{(1)} r^{n+1} + \beta_n A_1 p^n,$$

$$T_n^{(1)} = (I - \beta_n A_1)(I + \alpha A_1)^{-1} = I - (\alpha + \beta_n)\tilde{A}_1,$$

$$\tilde{A}_1 = A_1(I + \alpha A_1)^{-1},$$

$$T_n^{(2)} = (I - \alpha A_2)(I + \beta_n A_2)^{-1} = I - (\alpha + \beta_n)\tilde{A}_2, \tilde{A}_2 =$$

$$= A_2(I + \beta_n A_2)^{-1},$$

$$r^{n+1} = \tilde{T}_n^{(2)} \tilde{T}_n^{(1)} r^0 = \tilde{T}_n^{(2)} \tilde{r}^{(1)} = \tilde{T}_n^{(1)} r^0,$$

$$\tilde{T}_n^{(l)} = \prod_{k=0}^{n} T_n^{(l)},\ l = 1, 2, \quad \beta_k = \mu_k - \alpha$$

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} = f(x,y),$$

$$(x,y) \in \Omega,\ \ u|_\Gamma = g(x,y),\ \ \Gamma = \bar{\Omega}/\Omega,$$

$$(A_1 u)_{i,j} = e^{(1)} u_{i,j} - a\, u_{i-1,j} - c\, u_{i+1,j},$$

$$(A_2 u)_{i,j} = e^{(2)} u_{i,j} - b\, u_{i,j-1} - d\, u_{i,j+1},$$

$$i = 1, ..., N_1;\ \ j = 1, ..., N_2,$$

$$a = e^{-ph_x/2}/h_x^2,\ \ c = e^{ph_x/2}/h_x^2,\ \ e^{(1)} = a + c,$$

$$b = e^{-qh_y/2}/h_y^2,\ \ d = e^{qh_y/2}/h_y^2,\ \ e^{(2)} = b + d.$$

$$A_1 = \{a_{i,j} = \textbf{3-diag}\{-a, e^{(1)}, -c\}, \quad a_{i,j} \neq a_{j,i},$$

$$\hat{A}_1 = D\, A_1, \quad D = \textbf{diag}\{d_i\},\ \hat{A}_1 = \tilde{A}_1^T,$$

$$d_1 = 1, \quad d_i = d_{i-1}\gamma = \gamma^{i-1}, \quad \gamma = c/a, \quad i = 1, ..., N_1,$$

$$\hat{A}_1 = \{\hat{a}_{i,j} = a_{i,j}\gamma^{i-1}\},$$

$$\hat{A}_1 = D\, A_1, \quad D = \textbf{diag}\{d_i\}$$

$N_1 \setminus q$	0	1	2	4	16
31	**11** $8.69 \cdot 10^{-9}$	**16** $1.59 \cdot 10^{-9}$	**16** $1.67 \cdot 10^{-9}$	**16** $1.81 \cdot 10^{-9}$	**16** $2.18 \cdot 10^{-9}$
63	**17** $3.67 \cdot 10^{-10}$	**25** $4.11 \cdot 10^{-9}$	**25** $4.22 \cdot 10^{-9}$	**25** $4.41 \cdot 10^{-9}$	**25** $4.40 \cdot 10^{-9}$
127	**25** $9.04 \cdot 10^{-9}$	**38** $2.47 \cdot 10^{-8}$	**38** $2.50 \cdot 10^{-8}$	**38** $2.56 \cdot 10^{-8}$	**39** $9.75 \cdot 10^{-9}$

Table 1. The numbers of iteration and error for ADI method with polinomial Krylovs subspaces (14)

$N_1 \setminus q$	**8**	**12**	**16**	**20**	**24**	**28**
31	**11(12)**	**10(17)**	**9(18)**	**9(18)**	**8(16)**	**8(16)**
	$1.8 \cdot 10^{-8}$	$1.1 \cdot 10^{-8}$	$4.1 \cdot 10^{-9}$	$7.3 \cdot 10^{-8}$	$1.4 \cdot 10^{-7}$	$1.6 \cdot 10^{-7}$
	14(22)	**13(23)**	**12(24)**	**11(22)**	**11(22)**	**10(20)**
	$1.3 \cdot 10^{-8}$	$1.3 \cdot 10^{-8}$	$1.5 \cdot 10^{-9}$	$7.5 \cdot 10^{-8}$	$1.8 \cdot 10^{-7}$	$3.1 \cdot 10^{-6}$
63	**16(17)**	**14(24)**	**13(24)**	**9(18)**	**12(24)**	**11(22)**
	$2.1 \cdot 10^{-7}$	$7.2 \cdot 10^{-9}$	$1.4 \cdot 10^{-8}$	$7.3 \cdot 10^{-8}$	$2.4 \cdot 10^{-5}$	$5.2 \cdot 10^{-6}$
	24(43)	**20(36)**	**19(32)**	**11(22)**	**16(32)**	**15(30)**
	$2.7 \cdot 10^{-9}$	$2.1 \cdot 10^{-9}$	$1.0 \cdot 10^{-8}$	$7.5 \cdot 10^{-8}$	$1.3 \cdot 10^{-7}$	$3.4 \cdot 10^{-7}$
127	**26(49)**	**23(51)**	**21(42)**	**19(38)**	**18(36)**	**17(34)**
	$9.0 \cdot 10^{-8}$	$2.6 \cdot 10^{-9}$	$2.6 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$3.7 \cdot 10^{-2}$	$4.2 \cdot 10^{-2}$
	41(82)	**35(70)**	**31(59)**	**29(54)**	**27(50)**	**25(47)**
	0.13	$8.1 \cdot 10^{-4}$	$9.8 \cdot 10^{-9}$	$7.2 \cdot 10^{-10}$	$6.9 \cdot 10^{-10}$	$3.0 \cdot 10^{-9}$

Table 2. The results for ADI method (27) with "explicit" rational Krylov subspaces