Thin Layer Models for Geophysical Applications

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Second French-Russian Conference
Novosibirsk, September 25, 2014

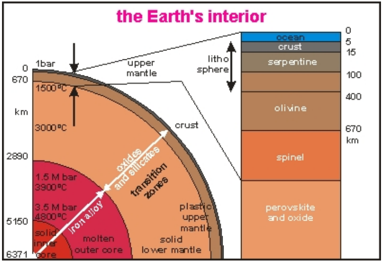
Motivations

HPC-GA Project

- **High Performance Computing for Geophysical Applications**
  FP7 Project : INRIA, BCAM, UFRGS, UNAM
  - Earthquakes
  - Tsunami
- **Numerical Models (WP2)**
  - Asymptotic method : To derive approximate models for the propagation of seismic waves and acoustic waves
  - Elasto-acoustic coupling : To reproduce earthquakes

Motivations

Configurations of interest

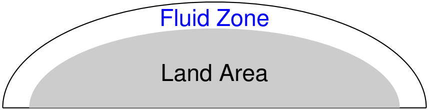


- **Configuration A** : The medium of interest is a Land Area surrounded by a fluid zone
- **Configuration B** : The medium of interest is a Land Area surrounded by a fluid zone and a part of the atmosphere

Motivations

Configuration A

- **Configuration A**:

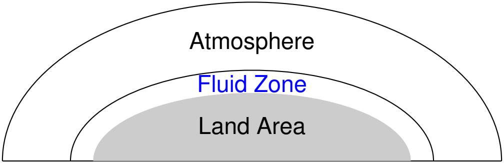


- The thickness of the fluid zone is negligible w.r.t. the wavelength
- **Sub-Configurations**:
  1. **Configuration A1** : A thin layer with a **uniform** thickness
  2. **Configuration A2** : A thin layer with a **variable** thickness

Motivations

Configuration B

- **Configuration B**



- **Difficulty**

  Apply a FEM on a mesh with thin cells in the Fluid Zone and much larger outside

Approach : an Asymptotic Method

1. "Replace" the thin layer by an Equivalent Boundary Condition
2. Couple this condition with
   - the elastic equation (**Configuration A**)
   - elastic and acoustic equations (**Configuration B**)

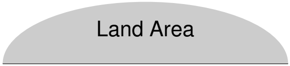


Figure: Configuration A without the thin layer

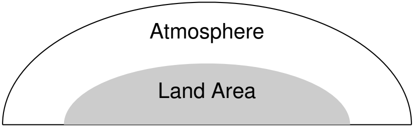


Figure: Configuration B without the thin layer

3. Apply a (DG) Finite Element Method

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- Derivation of Equivalent Conditions for thin layer Problems :

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DE CASTRO, DIAZ, PÉRON, 2014

Outline

Outline

A Model Problem

The Problem $(\mathbf{P}_\varepsilon)$ set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

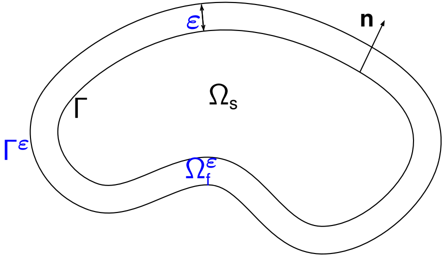
$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in} \quad \Omega_f^\varepsilon$$

$$\partial_{\mathbf{n}} p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on} \quad \Gamma$$

$$\underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = -p\mathbf{n} \quad \text{on} \quad \Gamma$$

$$p = 0 \quad \text{on} \quad \Gamma^\varepsilon$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of $(\mathbf{P}_\varepsilon)$ as $\varepsilon \to 0$?

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\,\underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

(i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$

(ii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ

(iii) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\,\underline{\underline{\epsilon}}(\mathbf{u})$

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(ii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ

(iii) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Assumption (SA)

The angular frequency ω *is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in} \quad \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = 0 & \text{on} \quad \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$*, the problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1_{0,\Gamma^\varepsilon}(\Omega_f^\varepsilon)$*, and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_f^\varepsilon} \leqslant C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ as $\varepsilon \to 0$

Uniform Estimates

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$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathrm{s}} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_\mathrm{f}^\varepsilon} \leqslant C\|\mathbf{f}\|_{0,\Omega_\mathrm{s}}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ as $\varepsilon \to 0$

Key for the proof:

To introduce a ”Scaled Problem” by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where} \quad s \in (0, \varepsilon)$$

Outline

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ when $\varepsilon \to 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \cdots$$
$$\mathrm{p}_\varepsilon(x) = \mathfrak{p}_0(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon \mathfrak{p}_1(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon^2 \mathfrak{p}_2(y_\alpha, \frac{s}{\varepsilon}) + \cdots$$

- **Step 2** : Equivalent Conditions of order $k+1 \in \mathbb{N}$.
  Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1})$$

- **Step 3** : Prove ”Stability” & Convergence results for Equivalent models

Step 1 : Multiscale Expansion - First terms

1. $$\mathfrak{p}_0 = 0$$

2. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on} \quad \Gamma \end{cases}$$

3. $$\mathfrak{p}_1(., S) = (S-1)\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,, \quad S \in (0,1)$$

4. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

5. $$\mathfrak{p}_2(., S) = -(S^2 - 1)\mathcal{H}\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} + (S-1)\rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

6. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} - \mathcal{H}\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\varepsilon^N := \mathbf{u}_\varepsilon - \sum_{n=0}^{N} \varepsilon^n \mathbf{u}_n \quad \text{in} \quad \Omega_s , \quad \text{and} \quad r_\varepsilon^N := \mathrm{p}_\varepsilon - \sum_{n=0}^{N} \varepsilon^n \mathrm{p}_n \quad \text{in} \quad \Omega_f^\varepsilon .$$

Evaluation of the right hand sides in

$$\begin{cases} \Delta r_\varepsilon^N + \kappa^2 r_\varepsilon^N & = & f_\varepsilon & \text{in} & \Omega_f^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\varepsilon^N) + \omega^2 \rho \mathbf{r}_\varepsilon^N & = & 0 & \text{in} & \Omega_s \\ \partial_{\mathbf{n}} r_\varepsilon^N - \rho_f \omega^2 \mathbf{r}_\varepsilon^N \cdot \mathbf{n} & = & g_\varepsilon & \text{on} & \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^N) + r_\varepsilon^N \mathbf{n} & = & 0 & \text{on} & \Gamma \\ r_\varepsilon^N & = & 0 & \text{on} & \Gamma^\varepsilon . \end{cases}$$

The RHS are explicit : $f_\varepsilon = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$ in Ω_f^ε , $\quad g_\varepsilon = \mathcal{O}(\varepsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_s} + \sqrt{\varepsilon}\|r_\varepsilon^N\|_{1,\Omega_f^\varepsilon} \leqslant C_N \varepsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2, 3\}$, we identify a simpler problem

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\varepsilon}^{k}) + \omega^2 \rho\, \mathbf{u}_{\varepsilon}^{k} = \mathbf{f} & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathrm{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n})\, \mathbf{n} = 0 & \text{on} \quad \Gamma \end{cases}$$

with $\mathrm{B}_{k,\varepsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

1. Order 1 :

$$\mathbf{T}(\mathbf{u}_0) = 0$$

2. Order 2 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^1) - \varepsilon\omega^2\rho_f\mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n}\,\mathbf{n} = 0$$

3. Order 3 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^2) - \varepsilon\omega^2\rho_f\left(1 - \varepsilon\mathcal{H}\right)\mathbf{u}_{\varepsilon}^2 \cdot \mathbf{n}\,\mathbf{n} = 0$$

4. Order 4 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^3) - \varepsilon\omega^2\rho_f\left(1 - \varepsilon\mathcal{H} + \frac{\varepsilon^2}{3}\left[\Delta_\Gamma + \kappa^2\mathbb{I} + 4\mathcal{H}^2 - \mathcal{K}\right]\right)\mathbf{u}_{\varepsilon}^3 \cdot \mathbf{n}\,\mathbf{n} = 0$$

Step 3 : Stability and Convergence results

$\mathrm{V}_k = \mathrm{H}^1(\Omega_s)$ when $k = 0, 1, 2$
$\mathrm{V}_k = \{\mathbf{u} \in \mathrm{H}^1(\Omega_s) \mid \ \mathbf{u}\cdot\mathbf{n}|_\Gamma \in \mathrm{H}^1(\Gamma)\}$ when $k = 3$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in \mathrm{L}^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathrm{V}_k$ and

- *Stability*
$$\|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\|\mathbf{f}\|_{0,\Omega_s}$$
- *Convergence*
$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1}$$

The case of a Fourier-Robin B.C.

Step 2 : Equivalent Conditions

When considering a Fourier-Robin boundary condition :

$$\partial_{\mathbf{n}} \mathrm{p}_{\varepsilon} - i\kappa \mathrm{p}_{\varepsilon} = 0 \quad \text{on} \quad \Gamma^{\varepsilon}$$

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\, \mathbf{n} = 0 \quad \text{on} \quad \Gamma$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^1) - i\omega c_f \rho_{\mathrm{f}} \left(1 + \varepsilon \left(-2\mathcal{H} + i\kappa^{-1}\Delta_{\Gamma}\right)\right) (\mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n})\, \mathbf{n} = 0$$

Outline

Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = \mathrm{D}(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($\mathbf{f} = 0$) : $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$-finite elements (Lagrange) in the Library Hou10ni
- Ω_s : Isotropic elastic material with Lamé coefficients :

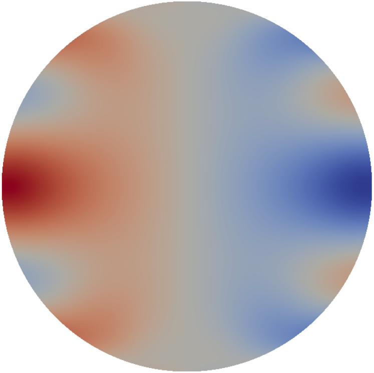
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 \,.$$

- $c = 1500 \text{ m.s}^{-1}, \quad \rho_f = 1000 \text{ kg.m}^{-3}\,, \quad \rho = 2700 \text{ kg.m}^{-3}.$

Numerical Simulations

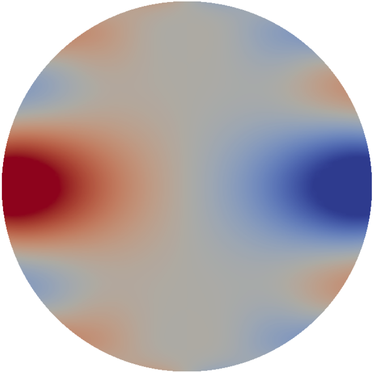
Qualitative Comparisons

$\varepsilon = 0.001;$



$|\mathrm{Re}\, \mathbf{u}_\varepsilon|$

$\mathbf{u}_\varepsilon$: analytical solution



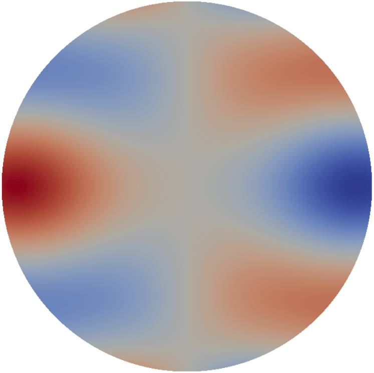
$|\mathrm{Re}\, \mathbf{u}^2_\varepsilon|$

$\mathbf{u}^2_\varepsilon$ (model of order 3) : IPDGM

Numerical Simulations

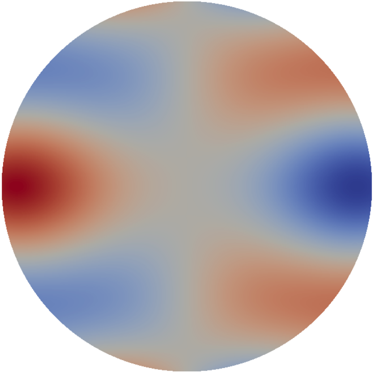
Qualitative Comparisons

$\varepsilon = 0.0001$;



$\left| \text{Re}\, \mathbf{u}_{\varepsilon} \right|$

$\mathbf{u}_{\varepsilon}$: analytical solution



$\left| \text{Re}\, \mathbf{u}_{\varepsilon}^{2} \right|$

$\mathbf{u}_{\varepsilon}^{2}$ (model of order 3) : IPDGM

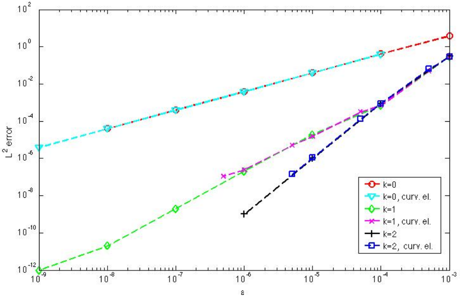
Convergence of the models

Dirichlet case

For $k \in \{0, 1, 2\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution or numerical solution (IPDGM)



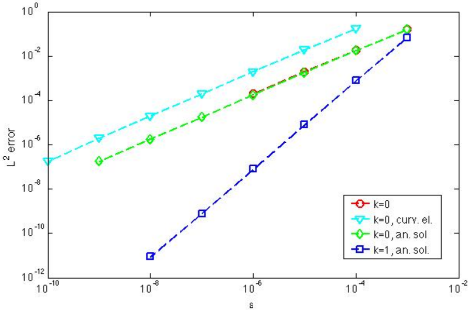
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution or numerical solution (IPDGM)



Outline

Configuration A2

Framework in 2D

- Parameterization of the layer $\Omega_{\mathrm{f}}^{\varepsilon}$

$$\Omega_{\mathrm{f}}^{\varepsilon} = \left\{ \mathbf{x}(t) + s\, f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \quad \mathbf{x}(t) \in \Gamma, \quad s \in (0, \varepsilon) \right\}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

Configuration A2

Framework in 2D

- Parameterization of the layer $\Omega_{\mathrm{f}}^{\varepsilon}$

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t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

- Euclidean metric of $\Omega_{\mathrm{f}}^{\varepsilon}$ defined in the coordinates (t, s):

$$\begin{pmatrix} \left(1 + s\, f(t)\mathrm{c}(t)\right)^2 + \left(s\, f'(t)\right)^2 & s\, f(t)f'(t) \\ s f(t)f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon f(t)\omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = 0$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon f(t)\omega^2 \rho_f \left(1 - \frac{\varepsilon}{2} f(t)\mathrm{c}(t)\right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = 0$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f \left(1 + \varepsilon\left(-f(t)\mathrm{c}(t) + i\kappa^{-1}[g(t)\partial_t + f(t)\partial_t^2]\right)\right)\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = 0$$

with

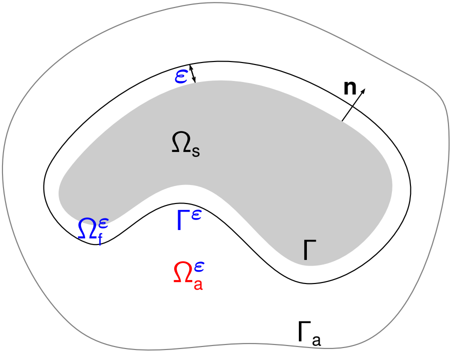
$$g(t) = \left(2 - f^2(t)\right) f'(t)$$

Configuration B

A Model Problem

$(\mathbf{u}, p, p^a)$ satisfies

$$\left\{\begin{array}{lll}
\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in} & \Omega_s \\
\Delta p + \kappa^2 p = 0 & \text{in} & \Omega_f^\varepsilon \\
\Delta p^a + \kappa_a^2 p^a = 0 & \text{in} & \Omega_a^\varepsilon \\
\partial_{\mathbf{n}} p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} & \text{on} & \Gamma \\
\underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = -p\mathbf{n} & \text{on} & \Gamma \\
p^a = p & \text{on} & \Gamma^\varepsilon \\
c_a^2 \partial_{\mathbf{n}} p^a = c_f^2 \partial_{\mathbf{n}} p & \text{on} & \Gamma^\varepsilon \\
\text{ABC} & \text{on} & \Gamma_a
\end{array}\right.$$



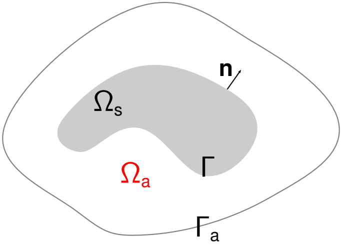
$$\kappa = \frac{\omega}{c_f}, \quad \kappa_a = \frac{\omega}{c_a}$$

Step 2 - Equivalent Conditions

Order 1

$(\mathbf{u}_0, p_0)$ satisfies

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_s \\ \Delta p_0^a + \kappa_a^2 p_0^a = 0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_0) = -p_0^a \mathbf{n} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} p_0^a = \left(\frac{c_f}{c_a}\right)^2 \rho_f \, \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$



Step 2 - Equivalent Conditions

Order 2

$(\mathbf{u}_1, p_1^a) := (\mathbf{u}_1^\varepsilon, p_1^\varepsilon)$ satisfies

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = \mathbf{f} & \text{in } \Omega_s \\ \Delta p_1^a + \kappa_a^2 p_1^a = 0 & \text{in } \Omega_a \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} = -p_1^a\, \mathbf{n} & \text{on } \Gamma \\ \partial_{\mathbf{n}} p_1^a + \varepsilon \left((\frac{c_f}{c_a})^2 - 1 \right) \Delta_\Gamma p_1^a = (\frac{c_f}{c_a})^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} & \text{on } \Gamma \\ \text{ABC} & \text{on } \Gamma_a \end{cases}$$

$$\theta_{1,\varepsilon} = \varepsilon \rho_f \omega^2 \left[1 - (\frac{c_f}{c_a})^2 \right]$$

$$\theta_{2,\varepsilon} = \rho_f \omega^2 - 2\varepsilon (1 - \rho_f \omega^2) \mathcal{H}$$

Prospect

1. To derive Equivalent Conditions adapted to high order ABCs
2. To implement Equivalent Conditions in configurations A2 and B
3. To implement Equivalent Conditions in 3D and in Time domain
4. To test these asymptotic models with real data

Thank you for your attention

Proof of Uniform Estimates

Variational Problem $(\mathrm{VP}_\varepsilon)$

- $V_\varepsilon = \{(\mathbf{u},\mathrm{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(\Omega_f^\varepsilon) \,|\, \gamma_0 \mathrm{p} = 0 \quad \text{on } \Gamma^\varepsilon\}$
- For all $\varepsilon > 0$, $(\mathbf{VP}_\varepsilon)$ writes : Find $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in V_\varepsilon$ such that

$$\forall (\mathbf{v},\mathrm{q}) \in V_\varepsilon \;, \qquad a_\varepsilon\left((\mathbf{u}_\varepsilon,\mathrm{p}_\varepsilon),(\mathbf{v},\mathrm{q})\right) = \langle F,(\mathbf{v},\mathrm{q})\rangle_{V_\varepsilon',V_\varepsilon}$$

$$a_\varepsilon\left((\mathbf{u},\mathrm{p}),(\mathbf{v},\mathrm{q})\right) := \int_{\Omega_f^\varepsilon} \left(\nabla \mathrm{p}\cdot\nabla\bar{\mathrm{q}} - \kappa^2 \mathrm{p}\bar{\mathrm{q}}\right)\,\mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_s} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}\cdot\bar{\mathbf{v}}\right) \mathrm{d}\mathbf{x} + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u}\cdot\mathbf{n}\,\bar{\mathrm{q}} + \mathrm{p}\bar{\mathbf{v}}\cdot\mathbf{n}\right)\,\mathrm{d}s$$

Formulation in a fixed domain

Scaled Problem $(\mathfrak{P}_\varepsilon)$

- $\Omega_f := \Gamma \times (0,1)$
- $V = \{(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(\Omega_f) \,|\, \mathfrak{p}(.\,,1) = 0 \quad \text{on}\, \Gamma\}$
- $(\mathfrak{P}_\varepsilon)$ writes : Find $(\mathbf{u}, \mathfrak{p}) \in V$ such that $\forall(\mathbf{v}, \mathfrak{q}) \in V$

$$\varepsilon \int_0^1 \int_\Gamma \left[(\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_\Gamma \mathfrak{p} \nabla_\Gamma \bar{\mathfrak{q}} + \varepsilon^{-2} \partial_S \mathfrak{p} \partial_S \bar{\mathfrak{q}} - \kappa^2 \mathfrak{p} \bar{\mathfrak{q}} \right] \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, \mathrm{d}\Omega_f$$

$$+ a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \, \bar{\mathfrak{q}} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n} \right) \, \mathrm{d}\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V',V}$$

Scaled Problem $(\mathfrak{P}_\varepsilon)$

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$*, the problem* $(\mathfrak{P}_\varepsilon)$ *with data* $\mathfrak{F}_\varepsilon \in V'$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ *which satisfies*

$$\sqrt{\varepsilon}\|\nabla_\Gamma \mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \sqrt{\varepsilon}^{-1}\|\partial_s \mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \|\mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \|\mathbf{u}_\varepsilon\|_{1,\Omega_s} \leqslant C\|\mathfrak{F}_\varepsilon\|_{V'} .$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$*, any solution* $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ *of problem* $(\mathfrak{P}_\varepsilon)$ *with a data* $\mathfrak{F}_\varepsilon \in V'$ *satisfies*

$$\|(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon)\|_{0,\Omega_s\times\Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_\varepsilon\|_{0,\Gamma} \leqslant C\|\mathfrak{F}_\varepsilon\|_{V'} .$$

Sketch of the proof of the Lemma

Assume : $\exists(\mathbf{u}_m, \mathfrak{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \to 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_{0,\Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_m\|_{0,\Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \to 0$$

We prove successively that :

1. $$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_W \leqslant C$$
   with $W = \{(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(0,1;\mathrm{L}^2(\Gamma)) \mid \mathfrak{p}(.\,,1) = 0 \quad \text{on}\,\Gamma\}$
2. $$\mathbf{u}_m \to \mathbf{u} \quad \text{in} \quad \mathrm{L}^2(\Omega_s) \quad \text{and} \quad \mathfrak{p}_m \to \mathfrak{p} = 0 \quad \text{in} \quad \mathrm{L}^2(\Omega_f)$$
3. $$\|\mathbf{u}\|_{0,\Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1$$
4. $\mathbf{u} = 0$ (using (SA)) : Contradiction.

Step 3 : Proof of Convergence

(i) Derive an expansion of $\mathbf{u}_\varepsilon^k$ and show that :

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k \ ,$$
$$\mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k \ .$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on} \quad \Gamma \ ,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1} \ .$$